Effects of Parametric Constraints on the CRLB in Gain and Phase Estimation Problems

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***Abstract*— The problem of estimating the direction-independent gain and phase characteristics of sensor arrays requires a boundary condition to solve the phase ambiguity of the solution. It has become common practice to use the constraint that the phase of the first element of the array is zero. By CRLB analysis we show analytically for calibration on a single point source of an array of identical elements that the variance of the phase estimates decreases by a factor $\frac{2p}{p-1}$ under the boundary condition that the sum of all phases equals zero where p is the number of elements in the array. We also show that this constraint is the one that among all possible constraints minimizes the total variance on all estimated parameters. Our analysis suggests that this conclusion also holds for arbitrary source models and arrays of non-identical elements. This statement is confirmed by repeating the CRLB analysis in simulation showing that the CRLB under this constraint coincides within the numerical accuracy with the minimal CRLB.**

***Index Terms*— Cramér-Rao Lower Bound, sensor arrays, calibration, gain estimation, phase estimation**

I. Introduction

The problem of estimating the direction-independent gain and phase characteristics of sensor arrays requires a boundary condition to obtain a unique solution. The physical reason for this is that the phase of an element within the array can only be determined with respect to the phase at some reference point. As a result the unconstrained solution to this estimation problem will produce a phase solution which correctly predicts the phase differences between the elements but has an arbitrary phase offset.

It has become common practice to impose the constraint that the first element of the array has zero phase as illustrated in [1]–[6]. In mathematical derivations this choice can be made without loss of generality and in some practical cases such a choice is required due to the necessity of a well defined phase reference.

In this paper analytic expressions will be derived for the Cramér-Rao lower bound (CRLB) for the gain and phase estimation problem when calibrating an array of identical elements on a single point source under the constraint that the first element has zero phase and under the constraint that the average phase of all elements is zero. We will show that under the latter constraint the CRLB for the phase estimate decreases by a factor $\frac{2p}{p-1}$ as compared to the first constraint where p is the number of elements in the array. It will then be shown that the proposed constraint does not only give a lower CRLB but is actually the constraint that minimizes the CRLB. We subsequently show that this constraint is also optimal in the case of an arbitrary source model and arrays of non-identical elements. Finally we will confirm our analysis by a simulation focusing on the calibration problem of a phased array radio telescope.

Notation: The transpose operator is denoted by $(\cdot)^T$, the complex conjugate (Hermitian) transpose by $(\cdot)^H$, the pseudo-inverse by $(\cdot)^\dagger$ and complex conjugation by $\bar{\cdot}$. An estimated value is denoted by $\hat{\cdot}$ and an expected value by $\mathcal{E}\{\cdot\}$. $\odot$ is the element-wise matrix multiplication (Hadamard product), $\mathrm{diag}(\cdot)$ converts a vector to a diagonal matrix with the vector placed on the main diagonal or converts the main diagonal of a square matrix to a column vector.

II. Data model

Let the output signal of the i^{th} element be denoted by $x_i(t)$ and define the array signal vector $\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_p(t)]^T$. We assume the presence of q mutually independent i.i.d. Gaussian signals $s_k(t)$ with variance σ_{s_k} impinging on the array, which are stacked in a $q \times 1$ vector $\mathbf{s}(t)$. Likewise, the sensor noise signals $n_i(t)$ are assumed to be mutually independent i.i.d. Gaussian signals with variance σ_{n_i} and are stacked in a $p \times 1$ vector $\mathbf{n}(t)$. If the narrow band condition holds, we can define the q spatial signature vectors $\mathbf{a}_k$, which include the phase delays due to the geometry and the directional response of the antennas. They are assumed to be known. The direction-independent element gains and phases which have to be calibrated can be described as $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, \cdots \gamma_p]^T$ and $\boldsymbol{\phi} = [\mathrm{e}^{\mathrm{j}\phi_1}, \mathrm{e}^{\mathrm{j}\phi_2}, \cdots, \mathrm{e}^{\mathrm{j}\phi_p}]^T$, respectively, with corresponding diagonal matrix forms $\boldsymbol{\Gamma} = \mathrm{diag}(\boldsymbol{\gamma})$ and $\boldsymbol{\Phi} = \mathrm{diag}(\boldsymbol{\phi})$. With these definitions, the array signal vector can be described as

$$\mathbf{x}(t) = \boldsymbol{\Gamma}\boldsymbol{\Phi}\left(\sum_{k=1}^{q} \mathbf{a}_k s_k(t)\right) + \mathbf{n}(t) = \boldsymbol{\Gamma}\boldsymbol{\Phi}\mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad (1)$$

where $\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_q]$ (size $p \times q$).

The signal is sampled with period T and N sample vectors are stacked in a data matrix $\mathbf{X} = [\mathbf{x}(T), \mathbf{x}(2T), \cdots, \mathbf{x}(NT)]$. The covariance matrix of $\mathbf{x}(t)$ is $\mathbf{R} = \mathcal{E}\left\{\mathbf{x}(t)\mathbf{x}^H(t)\right\}$ and is estimated by $\widehat{\mathbf{R}} = N^{-1}\mathbf{X}\mathbf{X}^H$. Likewise, the source signal covariance $\boldsymbol{\Sigma}_s = \mathrm{diag}(\boldsymbol{\sigma}_s)$ where $\boldsymbol{\sigma}_s = [\sigma_{s_1}, \sigma_{s_2}, \cdots \sigma_{s_q}]^T$, and the noise covariance matrix is $\boldsymbol{\Sigma}_n = \mathrm{diag}(\boldsymbol{\sigma}_n)$ where

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$\boldsymbol{\sigma}_n = [\sigma_{n_1}, \sigma_{n_2}, \cdots \sigma_{n_p}]^T$. Then the model for $\mathbf{R}$ based on (1) is

$$\mathbf{R} = \boldsymbol{\Gamma}\boldsymbol{\Phi}\mathbf{A}\boldsymbol{\Sigma}_s\mathbf{A}^H\boldsymbol{\Phi}^H\boldsymbol{\Gamma}^H + \boldsymbol{\Sigma}_n \,. \tag{2}$$

In this model, $\mathbf{A}$ and $\boldsymbol{\Sigma}_s$ are assumed to be known from tables. Algorithms to estimate $\boldsymbol{\Gamma}$, $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}_n$ are e.g., discussed in [1]–[5]. Although not all algorithms require this, we will assume here that $\boldsymbol{\Sigma}_n$ is known as well.

III. Cramér-Rao lower bound

We consider the estimation of a real-valued $n \times 1$ parameter vector $\boldsymbol{\theta}$. Stoica and Ng [7] have shown that if the estimate $\widehat{\boldsymbol{\theta}}$ is subject to k continuously differentiable constraints $\mathbf{f}(\widehat{\boldsymbol{\theta}}) = \mathbf{0}$, the CRLB has the form

$$\mathcal{E}\left\{\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)^T\right\} \geq \mathbf{U}\left(\mathbf{U}^T\mathbf{J}\mathbf{U}\right)^{-1}\mathbf{U}^T. \tag{3}$$

In Eq. (3), $\mathbf{J}$ denotes the Fisher Information Matrix (FIM) for the unconstrained parameter estimation problem and $\mathbf{U}$ is an $n \times (n-k)$ matrix whose columns form an orthonormal basis for the null space of the $k \times n$ gradient matrix of $\mathbf{f}(\widehat{\boldsymbol{\theta}})$,

$$\mathbf{F}(\boldsymbol{\theta}) = \frac{\delta \mathbf{f}(\boldsymbol{\theta})}{\delta \boldsymbol{\theta}^T}. \tag{4}$$

This implies that $\mathbf{F}(\boldsymbol{\theta})\mathbf{U} = 0$ while $\mathbf{U}^T\mathbf{U} = \mathbf{I}$. Earlier, Gorman and Hero [8] have found a similar expression for the CRLB under parametric constraints using a derivation based on the Barankin bound.

For our application, the complex gain estimation problem for sensor arrays can be parameterized in terms of gain amplitudes and gain phases. This representation gives

$$\boldsymbol{\theta} = [\gamma_1, \gamma_2, \cdots, \gamma_p \mid \phi_1, \phi_2, \cdots, \phi_p]^T \,. \tag{5}$$

The aim of this analysis is to compare the CRLB under the following two parametric constraints:

$$\mathbf{f}_1(\boldsymbol{\theta}) = \phi_1 = 0 \tag{6}$$

$$\mathbf{f}_2(\boldsymbol{\theta}) = \sum_{i=1}^{p} \phi_i = 0 \tag{7}$$

The gradient matrices for these constraints are

$$\mathbf{F}_1(\boldsymbol{\theta}) = [\mathbf{0}_p^T \mid 1, \mathbf{0}_{p-1}^T] \tag{8}$$

$$\mathbf{F}_2(\boldsymbol{\theta}) = [\mathbf{0}_p^T \mid \mathbf{1}_p^T] \tag{9}$$

where $\mathbf{0}_p$ and $\mathbf{1}_p$ denote column vectors of size $p \times 1$ filled with zeros and ones respectively. Orthonormal bases for the null spaces of these gradient matrices are

$$\mathbf{U}_1 = \left[\begin{array}{c|c} \mathbf{I}_p & \mathbf{0}_{p\times p-1} \\ \hline \mathbf{0}_{p\times p} & \begin{array}{c}\mathbf{0}_{p-1}^T \\ \mathbf{I}_{p-1}\end{array} \end{array}\right] \tag{10}$$

$$\mathbf{U}_2 = \left[\begin{array}{c|c} \begin{array}{c}\mathbf{0}_{p-1}^T \\ \mathbf{I}_{p-1}\end{array} & \begin{array}{c}\mathbf{1}_p^T/\sqrt{p} \\ \mathbf{0}_{p-1\times p}\end{array} \\ \hline \mathbf{0}_{p\times p-1} & \mathbf{I}_p - \mathbf{1}_p\mathbf{1}_p^T/p \end{array}\right] \tag{11}$$

where $\mathbf{I}_p$ denotes the $p \times p$ identity matrix.

Let $\mathbf{R}_0 = \mathbf{A}\boldsymbol{\Sigma}_s\mathbf{A}^H$ and introduce

$$\mathbf{M}_1 = \left(\left(\boldsymbol{\Gamma}\overline{\boldsymbol{\Phi}\mathbf{R}_0}\boldsymbol{\Phi}\right)^H \overline{\mathbf{R}}^{-1}\left(\boldsymbol{\Gamma}\overline{\boldsymbol{\Phi}\mathbf{R}_0}\boldsymbol{\Phi}\right)\right) \odot \mathbf{R}^{-1} \tag{12}$$

$$\mathbf{M}_2 = \left(\left(\boldsymbol{\Gamma}\overline{\boldsymbol{\Phi}\mathbf{R}_0}\boldsymbol{\Phi}\right)^H \overline{\mathbf{R}}^{-1}\right) \odot \left(\mathbf{R}^{-1}\left(\boldsymbol{\Gamma}\boldsymbol{\Phi}\mathbf{R}_0\overline{\boldsymbol{\Phi}}\right)\right) \tag{13}$$

Following the procedure outlined in the appendix of [5], we find that the unconstrained FIM can then be expressed as

$$\mathbf{J} = \begin{bmatrix} 2\mathrm{Re}\{\mathbf{M}_1 + \mathbf{M}_2\} & 2\mathrm{Im}\{\mathbf{M}_2 - \mathbf{M}_1\}\boldsymbol{\Gamma} \\ 2\boldsymbol{\Gamma}\mathrm{Im}\{\mathbf{M}_2^H - \mathbf{M}_1^H\} & 2\boldsymbol{\Gamma}\mathrm{Re}\{\mathbf{M}_1 - \mathbf{M}_2\}\boldsymbol{\Gamma} \end{bmatrix}. \tag{14}$$

IV. Single point source

In this section we specialize to calibration based on a single point source; the case $q > 1$ is considered in Section V. Without loss of generality we can assume that the single source is positioned in the phase center of the array and has unit power. This simplifies the source covariance model to $\mathbf{R}_0 = \mathbf{1}_p\mathbf{1}_p^T$. Assume also that the array consists of identical elements, so we may take $\boldsymbol{\Gamma} = \mathbf{I}$, $\boldsymbol{\Phi} = \mathbf{I}$ and $\boldsymbol{\Sigma}_n = \sigma\mathbf{I}$ where σ is the noise power of a single element. The covariance matrix model now simplifies to $\mathbf{R} = \mathbf{1}_p\mathbf{1}_p^T + \sigma\mathbf{I}$ which can be inverted using the relation

$$\left(\mathbf{B} + \mathbf{b}\mathbf{b}^H\right)^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\mathbf{b}\mathbf{b}^H\mathbf{B}^{-1}}{1 + \mathbf{b}^H\mathbf{B}^{-1}\mathbf{b}}. \tag{15}$$

This yields

$$\mathbf{R}^{-1} = \frac{1}{\sigma}\left(\mathbf{I} - \frac{1}{\sigma + p}\mathbf{1}_p\mathbf{1}_p^T\right) \tag{16}$$

Using these simplifications, Eqs. (12) and (13) reduce to

$$\mathbf{M}_1 = \frac{p}{\sigma(\sigma + p)}\left(\mathbf{I} - \frac{1}{\sigma + p}\mathbf{1}_p\mathbf{1}_p^T\right) \tag{17}$$

$$\mathbf{M}_2 = \frac{1}{(\sigma + p)^2}\mathbf{1}_p\mathbf{1}_p^T. \tag{18}$$

By inserting these results in Eq. (14) one finds

$$\mathbf{J} = \frac{2}{\sigma(\sigma + p)}\begin{bmatrix} p\mathbf{I} + \frac{\sigma - p}{\sigma + p}\mathbf{1}_p\mathbf{1}_p^T & \mathbf{0}_{p\times p} \\ \mathbf{0}_{p\times p} & p\mathbf{I} - \mathbf{1}_p\mathbf{1}_p^T \end{bmatrix}. \tag{19}$$

Note (as expected) that $\mathbf{J}$ is singular with a 1-dimensional null space, spanned by the vector

$$\mathbf{v}_0 = \frac{1}{\sqrt{p}}\begin{bmatrix} \mathbf{0}_p \\ \mathbf{1}_p \end{bmatrix} \tag{20}$$

Thus, a constraint to avoid this singularity is necessary.

Substitution of Eqs. (10) and (19) in Eq. (3) gives the CRLB under the constraint $\phi_1 = 0$ as

$$E\left(\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)^T\right) \geq \frac{\sigma(\sigma + p)}{2p} \times$$
$$\times \left[\begin{array}{c|c} \mathbf{I}_p - \frac{\sigma - p}{2\sigma p}\mathbf{1}_p\mathbf{1}_p^T & \mathbf{0}_{p\times p} \\ \hline \mathbf{0}_{p\times p} & \begin{array}{cc} 0 & \mathbf{0}_{p-1}^T \\ \mathbf{0}_{p-1} & \mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}^T \end{array} \end{array}\right]. \tag{21}$$

This means that

$$\mathrm{var}\left(\gamma_i\right) = \frac{\sigma(\sigma + p)}{2p}\left(1 - \frac{\sigma - p}{2\sigma p}\right) \tag{22}$$

for all elements and that

$$\mathrm{var}\left(\phi_i\right) = \frac{\sigma(\sigma+p)}{p} \tag{23}$$

for all elements except the first. The variance of the phase of the first element is zero, since $\phi_1 = 0$ by definition.

The CRLB under the constraint $\sum_{i=1}^{p} \phi_i = 0$ can be found in a similar way using Eqs. (3), (11) and (19) as

$$\begin{aligned} & E\left(\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)^T\right) \geq \\ & \frac{\sigma(\sigma+p)}{2p}\begin{bmatrix} \mathbf{I}_p - \frac{\sigma-p}{2\sigma p}\mathbf{1}_p\mathbf{1}_p^T & \mathbf{0}_{p\times p} \\ \mathbf{0}_{p\times p} & \mathbf{I}_p - \mathbf{1}_p\mathbf{1}_p^T/p \end{bmatrix}. \end{aligned} \tag{24}$$

This means that

$$\mathrm{var}\left(\gamma_i\right) = \frac{\sigma(\sigma+p)}{2p}\left(1 - \frac{\sigma-p}{2\sigma p}\right) \tag{25}$$

for all elements. This result is the same as the result obtained under the constraint $\phi_1 = 0$. This outcome was expected since neither constraint affects the gain estimates. For the variance of the phase estimates the result is

$$\mathrm{var}\left(\phi_i\right) = \frac{\sigma(\sigma+p)}{2p}\left(1 - \frac{1}{p}\right) \tag{26}$$

which differs from the previous result by a factor $\frac{2p}{p-1}$. If p is reasonably large this factor is approximately equal to 2.

In [9], De Carvalho, Cioffi and Slock pose that among all sets of a minimal number of independent constraints *the pseudo-inverse of the unconstrained FIM* $\mathbf{J}^\dagger$ *yields the lowest value for the total variance on all estimated parameters.* An eigenvalue decomposition on $\mathbf{J}$ will yield $q = \mathrm{rank}(\mathbf{J})$ eigenvalues $\lambda_i \neq 0$ with corresponding eigenvectors $\mathbf{v}_i$. If $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q]$ and $\boldsymbol{\Lambda} = \mathrm{diag}([\lambda_1, \lambda_2, \cdots, \lambda_q])$, the pseudo-inverse can be computed as

$$\mathbf{J}^\dagger = \mathbf{V}\boldsymbol{\Lambda}^{-1}\mathbf{V}^T = \mathbf{V}\left(\mathbf{V}^T\mathbf{J}\mathbf{V}\right)^{-1}\mathbf{V}^T \tag{27}$$

which has the same form as Eq. (3). Therefore, to achieve the lowest total variance on all estimated parameters, we need to set $\mathbf{U} = \mathbf{V}$, or (more in general) select $\mathbf{U}$ such that the column span of $\mathbf{U}$ equals the column span of $\mathbf{V}$. In other words, the constraints should be selected such that its gradient matrix spans the orthogonal complement of $\mathbf{V}$, the range of the FIM. This complement space is described by the eigenvectors corresponding to the zero valued eigenvalues of the FIM.

In the case studied here, one eigenvalue of $\mathbf{J}$ in (14) is equal to zero, with corresponding eigenvector $\mathbf{v}_0 = (1/\sqrt{p})[\mathbf{0}_p^T; \mathbf{1}_p^T]^T$. This is precisely the space spanned by the gradient matrix $\mathbf{F}_2(\boldsymbol{\theta})$ in Eq. (9). This proves that $\sum_{i=1}^{p} \phi_1 = 0$ is a constraint that minimizes the total variance on the estimated parameters.

V. Multiple sources

When studying more complicated source models following the steps outlined in the previous section, one does not obtain a pedagogically interesting expression. We will show below, however, that $\mathbf{v}_0$ still lies in the null space of $\mathbf{J}$ in the general case with an arbitrary source covariance model $\mathbf{R}_0$ and arbitrary direction-independent element gains and phases, i.e. for $\boldsymbol{\Gamma} \neq \mathbf{I}$ and $\boldsymbol{\Phi} \neq \mathbf{I}$ suggesting that the constraint $\sum_{i=1}^{p} \phi_i = 0$ also minimizes the total variance on all parameters in the more general case.

We need to show that $\mathbf{J}\mathbf{v}_0 = \mathbf{0}$. Since $\mathbf{J}$ is a positive semi-definite matrix, it suffices to prove that $\mathbf{v}_0^T\mathbf{J}\mathbf{v}_0 = 0$. Using the structure of $\mathbf{v}_0$, this reduces to prove that, for the (2,2) block of $\mathbf{J}$,

$$\mathbf{1}_p^T \mathrm{Re}\left\{\boldsymbol{\Gamma}(\mathbf{M}_1 - \mathbf{M}_2)\boldsymbol{\Gamma}\right\}\mathbf{1}_p = 0. \tag{28}$$

As starting point we note that

$$\begin{aligned} \boldsymbol{\Gamma}\mathbf{M}_1\boldsymbol{\Gamma} &= \\ &= \left(\left(\overline{\boldsymbol{\Phi}\mathbf{R}_0}\boldsymbol{\Phi}\right)\left(\boldsymbol{\Gamma}\overline{\mathbf{R}}^{-1}\boldsymbol{\Gamma}\right)\left(\overline{\boldsymbol{\Phi}\mathbf{R}_0}\boldsymbol{\Phi}\right)\right) \odot \left(\boldsymbol{\Gamma}\mathbf{R}^{-1}\boldsymbol{\Gamma}\right) \\ &= \left(\overline{\mathbf{Z}\mathbf{Y}\mathbf{Z}}\right) \odot \mathbf{Y} \end{aligned} \tag{29}$$

where we have introduced the hermitian matrices $\mathbf{Y} = \boldsymbol{\Gamma}\mathbf{R}^{-1}\boldsymbol{\Gamma}$ and $\mathbf{Z} = \boldsymbol{\Phi}\mathbf{R}_0\boldsymbol{\Phi}^H$. In a similar way we find that

$$\boldsymbol{\Gamma}\mathbf{M}_2\boldsymbol{\Gamma} = \left(\overline{\mathbf{Z}\mathbf{Y}}\right) \odot \left(\mathbf{Y}\mathbf{Z}\right) \tag{30}$$

By inserting Eqs. (29) and (30) in Eq. (28) we obtain

$$\begin{aligned} & \mathbf{1}_p^T \mathrm{Re}\left\{\boldsymbol{\Gamma}(\mathbf{M}_1 - \mathbf{M}_2)\boldsymbol{\Gamma}\right\}\mathbf{1}_p = \\ &= \mathbf{1}_p^T \mathrm{Re}\left\{\left(\overline{\mathbf{Z}\mathbf{Y}\mathbf{Z}}\right) \odot \mathbf{Y} - \left(\overline{\mathbf{Z}\mathbf{Y}}\right) \odot \left(\mathbf{Y}\mathbf{Z}\right)\right\}\mathbf{1}_p \\ &= \mathrm{Re}\left\{\mathrm{diag}\left(\left(\overline{\mathbf{Z}\mathbf{Y}\mathbf{Z}}\right)^T\mathbf{Y} - \left(\overline{\mathbf{Z}\mathbf{Y}}\right)^T\mathbf{Y}\mathbf{Z}\right)^T\right\}\mathbf{1}_p \\ &= \mathrm{Re}\left\{\mathrm{trace}\left(\mathbf{Z}\mathbf{Y}\mathbf{Z}\mathbf{Y} - \left(\mathbf{Z}\mathbf{Y}\mathbf{Z}\mathbf{Y}\right)^H\right)\right\} \\ &= 0, \end{aligned} \tag{31}$$

which completes the proof.

VI. Application example

We illustrate the multi-source case by an example. The Low Frequency Array (LOFAR) is a phased array radio telescope currently under construction in The Netherlands. It will consist of 77 stations each consisting of 96 dual polarized low band antennas operating in the 10-90 MHz frequency range and 96 dual polarized high band antennas operating in the 110-250 MHz range. By the end of 2003 the Initial Test Station (ITS) consisting of $p = 60$ inverted V-shaped dipoles arranged in a 5-armed spiral configuration became operational [10].

For this example we will use the ITS antenna configuration and assume a sky model at 30 MHz consisting of the strongest $q = 10$ astronomical sources which were visible in the sky above the ITS on January 26, 2005 at midnight. The source locations and power ratios were taken from the third Cambridge catalog of radio sources [11] and the total power of these sources was assumed to be 1% of the system noise power of the individual antennas. An integration time of 4 s was assumed which corresponds to $N = 156250$ samples in a Nyquist sampled 40 kHz frequency channel.

We will compare the CRLBs (3) using each of the two constraints ($\phi_1 = 0$ and $\sum_{i=1}^{p} \phi_i = 0$), and also show the lower bound given by $\mathbf{J}^\dagger$. As parameter values, we take identical antennas, i.e., $\boldsymbol{\Gamma} = \mathbf{I}$ and $\boldsymbol{\Phi} = \mathbf{I}$. The result is shown in Fig. 1. The horizontal axis is the index of the parameter vector θ in (5). The vertical axis is the computed bound on the variance.

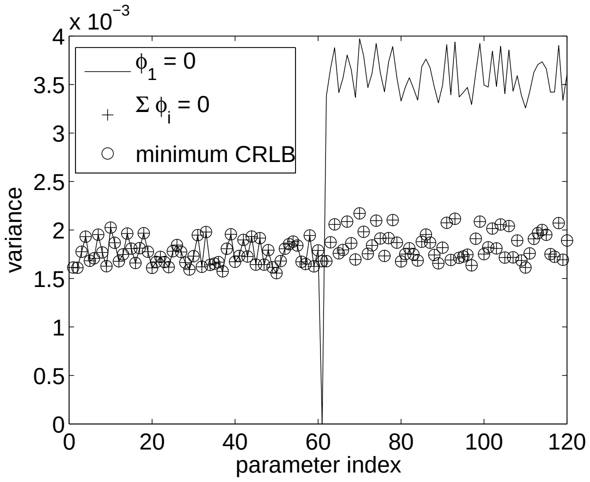

Fig. 1. This plot shows CRLBs for the multi source complex gain estimation problem presented in the text under the constraint $\phi_1 = 0$, $\sum_{i=1}^{p} \phi_i = 0$ and based on de pseudo-inverse of the FIM which should give the lowest total variance on the estimated parameters. Note that the latter coincides with the CRLB under the constraint $\sum_{i=1}^{p} \phi_i = 0$ indicating that this constraint leads to the lowest possible CRLB.

The plot shows that, as in the single source case, the constraints have no impact on the variance on the gain estimates and that the variance on the phase estimates decreases by approximately a factor 2. The plot also shows that the variance varies from one element to another. This is caused by two effects. First, the elements have different gains. The power picked up from the sources therefore differs for different elements. The receiver noise powers are assumed to be equal. Thus the signal-to-noise ratio is varying from one element to another. The second effect is interference between the source signals in the array covariance matrix. These interference effects are fully determined by the array geometry and the source model and cause variation in signal amplitudes in the crosscorrelations. Since all elements are assumed to have the same sensor noise power, the noise power is the same in all crosscorrelations. Thus some crosscorrelations have better signal-to-noise ratio than others.

In the previous section it was demonstrated that $\mathbf{v}_0$ lies in the null space of the FIM, suggesting that the constraint $\sum_{i=1}^{p} \phi_i = 0$ gives the same CRLB as the CRLB based on the pseudo-inverse of the FIM which gives the lowest total variance on the estimated parameters. In Fig. 1 we also plotted the latter CRLB for the multiple source situation outlined above. The CRLB under the constraint $\sum_{i=1}^{p} \phi_i = 0$ coincides within the numerical accuracy of the simulation with the CRLB based on the pseudo-inverse as expected based on the computations presented above.

VII. Concluding remarks

In this paper the CRLB for gain and phase estimation was studied under the constraints $\phi_1 = 0$ and $\sum_{i=1}^{p} \phi_i = 0$. It was found that for calibration on a single point source of an array of identical elements the variance on the phase estimates is larger by a factor $\frac{2p}{p-1}$ under the first constraint as compared to the latter, which is in fact the constraint that minimizes the total variance on the estimated parameters. It was also shown that similar conclusions hold for more general cases involving arbitrary source covariance models and arrays of non-identical elements.

We offer the following intuition for this loss by a factor of almost 2. Suppose that the phase parameter vector is estimated without a phase constraint. In that case all phases are shifted by an arbitrary offset which cannot be identified. An implementation of the first constraint would set $\phi_1 = 0$ by subtracting ϕ_1 from all other phase estimates. This sets the variance of the estimate of ϕ_1 equal to zero, but adds it to the phases of all other elements, thus doubling their variance. The second constraint would compute the average phase and subtract it from all elements, which leads to a small variance decrease. This intuition can be followed in many similar cases as well, e.g., in the context of blind source separation and equalization, where often the phase is not uniquely identifiable.

It can be shown that if the estimation problem can be reformulated in terms of $p-1$ phase differences, the variances of these $p-1$ phase parameters are the same under both constraints. This indicates that if the system uses only knowledge of the phase differences between the elements, the result is not affected by the choice for either of the two constraints discussed in this paper. The difference in the variances on the gain phases does matter if phase stability of the whole system is required. An example is a nonlinear transformation (e.g., converting the phases to complex beamformer weights), followed by averaging (e.g., as is done implicitly in an adaptive filter). In that case it is important to have a phase reference with lower variance.

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