2nd URSI AT-RASC, Gran Canaria, 28 May – 1 June 2018

Assessing Polarimetric Performance in Scenarios with Spatially Non-White Noise

Stefan J. Wijnholds
Netherlands Institute for Radio Astronomy, Dwingeloo, The Netherlands, wijnholds@astron.nl

Abstract

Current figures-of-merit to assess polarimetric performance of an antenna (array) are based on the far-field response of the antenna (array). This implicitly assumes that the noise is spatially white and that there is no noise coupling between the receiver paths in the system. In this paper, I demonstrate that this may have a profound effect on the capability of the antenna system to reconstruct the polarisation of a received signal and propose a new figure-of-merit to capture that reconstruction capability.

1 Introduction

Figures-of-merit (FoM) like cross-polarisation discrimination (XPD) [1], cross-polarisation isolation (XPI) [1] and intrinsic cross-polarisation ratio (IXR) [2] are commonly used to characterise the polarimetric performance of antenna systems. These FoMs measure the orthogonality of the far-field response of the antenna (array) to two incoming plane waves with perfectly orthogonal polarisation, i.e., these FoMs assess the response to an incoming signal without considering the noise in the individual receive paths of the antenna (array). This effectively means that the receiver noise is assumed to be spatially white. In many practical systems, this may not be true due to a spatially non-white distribution of background noise [3] or due to noise coupling [4, 5].

This may have a significant impact on the ability to reconstruct the polarimetric state of the field received by the system. Consider, for example, an antenna consisting of two ideal, orthogonally-oriented dipoles observing in the boresight direction. According to the standard FoMs mentioned above, this is a perfect antenna in terms of polarimetric performance. However, if the noise power in the receive path of one of the dipoles is a factor 10 higher than that in the other receive path, the sensitivity of this system to linear polarisation aligned with the first dipole is a factor 10 lower than that to the orthogonal direction of polarisation. This is obviously undesirable when one needs to be able to reconstruct the polarimetric state of incoming signals with, possibly, arbitrary polarisation. Even in radio astronomical observations aiming to detect weak sources, where we can usually assume that the incoming signal is unpolarised, it is desirable to receive both polarisation with (close to) equal sensitivity. This is due to the fact that one needs to assign weights proportional to the signal-to-noise (SNR) ratio to each received polarisation before adding them to maximise the SNR of an unpolarised source.

In this paper, I therefore propose a new FoM based on the sensitivity ratio of best and worst detectable polarimetric state. This FoM retains the attractive property of the IXR that it is independent of the choice of coordinate system used to describe the geometry of the antenna and polarisation state of the electromagnetic field.

2 System model

Following the conventions used in [6], the response of an array of antennas can be characterised by the voltage responses of all feeds to two sources with unit power and perfectly orthogonal polarisation, which can be denoted by u and v. These voltage responses can be stacked in vectors $\mathbf{v}_u$ and $\mathbf{v}_v$, which can be combined in a matrix $\mathbf{V} = [\mathbf{v}_u, \mathbf{v}_v]$. If the array is observing a single source with 2×2 source covariance matrix Σ, the array covariance matrix for the full array can be modeled as

$$\mathbf{R} = \mathbf{R}_\mathrm{s} + \mathbf{R}_\mathrm{n} = \mathbf{V}\Sigma\mathbf{V}^H + \mathbf{R}_\mathrm{n}, \quad (1)$$

where $\mathbf{R}_\mathrm{s}$ is the noise-free array covariance matrix in response to the external source and $\mathbf{R}_\mathrm{n}$ is the noise covariance matrix.

To get an intuition for this abstract formulation of the array response, consider an array of identical antennas, each consisting of two feeds, whose antenna response can be described by a Jones matrix $\mathbf{J}$. Assuming that the feed numbering is chosen such that the two feeds of each antenna have consecutive indices and that the geometrical delay between each antenna is the same (or perfectly compensated), the matrix $\mathbf{V}$ is described by $\mathbf{V} = \mathbf{1} \otimes \mathbf{J}$, where $\mathbf{1}$ denotes a vector with length equal to the number of antennas in the array containing only ones and $\otimes$ denotes the Kronecker product.

A polarimetric antenna array requires a beamformer with two output ports. The output signals at the two output ports are obtained by adding the individual feed signals with weights $\mathbf{w}_1$ and $\mathbf{w}_2$ respectively. After stacking those weights in a matrix $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2]$, the covariance of the two beamformer output signals can be described by

$$\begin{aligned}\mathbf{R}_\mathrm{o} &= \mathbf{W}^H\mathbf{R}\mathbf{W} \\ &= \mathbf{W}^H\mathbf{V}\Sigma\mathbf{V}^H\mathbf{W} + \mathbf{W}^H\mathbf{R}_\mathrm{n}\mathbf{W}. \qquad (2)\end{aligned}$$

3 Sensitivity and optimal beamforming

The sensitivity, measured as the ratio of effective area A_e and equivalent antenna noise temperature T_sys, of an antenna (array) with a single-polarisation beamformer using weight vector $\mathbf{w}$ is given by

$$\frac{A_\mathrm{e}}{T_\mathrm{sys}} = \frac{k_\mathrm{B}B}{S_\mathrm{s}}\frac{\mathbf{w}^H\mathbf{R}_\mathrm{s}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_n\mathbf{w}}, \qquad (3)$$

where k_B is the Boltzmann constant, B is the observed bandwidth and S_s is the power flux density of the received wavefront. The two-port beamformer introduced above can only construct a beam matched to a specific polarisation by forming a superposition of the two output ports, i.e., using weights $\mathbf{w} = \mathbf{W}\mathbf{a}$ where $\mathbf{a} = [a_1, a_2]^T$. As shown in the appendix of [6], we can introduce $\mathbf{a}' = \mathbf{C}^{-1/2}\mathbf{V}^H\mathbf{W}\mathbf{a}$ where

$$\mathbf{C} = \mathbf{V}^H\mathbf{W}\left(\mathbf{W}^H\mathbf{R}_\mathrm{n}\mathbf{W}\right)^{-1}\mathbf{W}^H\mathbf{V}. \qquad (4)$$

The matrix $\mathbf{C}$ represents the covariance matrix of the two beamformer output ports after whitening the noise at the beamformer output ports. As a result of this, we obtain

$$\frac{A_\mathrm{e}}{T_\mathrm{sys}} = \frac{k_\mathrm{B}B}{S_\mathrm{s}}\frac{\mathbf{a}'^H\mathbf{C}\mathbf{a}'}{\mathbf{a}'^H\mathbf{a}'}. \qquad (5)$$

This implies that the sensitivity of the two beamformer outputs to any specific polarisation state lies in the field-of-values [7] of the matrix $\mathbf{C}$. The sensitivity of the individual beamformer outputs is therefore bounded by

$$\frac{k_\mathrm{B}B}{S_\mathrm{s}}\lambda_\mathrm{min} \leq \frac{A_\mathrm{e}}{T_\mathrm{sys}} \leq \frac{k_\mathrm{B}B}{S_\mathrm{s}}\lambda_\mathrm{max}, \qquad (6)$$

where λ_min and λ_max are the smallest and largest eigenvalue of $\mathbf{C}$ respectively.

An optimal beamformer maximises the SNR of the source while perfectly reconstructing the polarisation state of the source. Mathematically, this can be formulated as

$$\underset{\mathbf{W}}{\mathrm{argmin}}\ \mathrm{Trace}\left\{\mathbf{W}^H\mathbf{R}_\mathrm{n}\mathbf{W}\right\} \ \text{ subject to } \ \mathbf{W}^H\mathbf{V} = \mathbf{I}, \qquad (7)$$

where $\mathbf{I}$ denotes the identity matrix. Solving for $\mathbf{W}$ gives the optimal beamformer weights

$$\mathbf{W}_\mathrm{opt} = \mathbf{R}_\mathrm{n}^{-1}\mathbf{V}\left(\mathbf{V}^H\mathbf{R}_\mathrm{n}^{-1}\mathbf{V}\right)^{-1}. \qquad (8)$$

Determination of the optimal beamforming weights thus requires knowledge of the far-field response of the feeds as described by $\mathbf{V}$ and the noise covariance matrix $\mathbf{R}_\mathrm{n}$. In the design phase of an instrument, these can be obtained from detailed electromagnetic simulations (see [8] for an example). Experimentally, the far-field responses can be obtained using holographic measurements [9] or using a test source [10], while the noise covariance matrix can be estimated by an off-source measurement, i.e., by observing an empty scene.

4 Sensitivity ratio FoM

The key features of the IXR as a FoM for polarimetric performance are that it provides an estimate of the possible magnification of errors (due to both instrumental modelling errors and noise) during reconstruction of the polarisation state of the incoming signal and that it is independent of the coordinate system chosen to describe the measurement. This is accomplished by defining the IXR in terms of the condition number of the Jones matrix $\kappa(\mathbf{J})$ describing the antenna response:

$$\mathrm{IXR} = \left(\frac{\kappa(\mathbf{J})+1}{\kappa(\mathbf{J})-1}\right)^2. \qquad (9)$$

The condition number of a matrix can be calculated by taking the ratio of its largest and smallest singular values. A physical interpretation of the condition number of the Jones matrix of an antenna is therefore the ratio of the maximum possible and minimum possible voltage gain provided by the antenna depending on the polarisation state of the received signal.

Based on Eq. (6), we can define, in a similar way, the ratio of the sensitivity to a signal in the most favourable polarisation state and the sensitivity to a signal in the least favourable polarisation state. This results in the Sensitivity Ratio (SR) FoM

$$\mathrm{SR} = \frac{\lambda_\mathrm{max}}{\lambda_\mathrm{min}}, \qquad (10)$$

where λ_max and λ_{min} are, respectively, the largest and smallest eigenvalues of $\mathbf{C}$ as defined in Eq. (6). Since $\mathbf{C}$ is a normal matrix and is positive semidefinite, the SR defined in Eq. (10) is equal to the condition number of $\mathbf{C}$.

To understand how the SR is related to $\kappa(\mathbf{J})$, and thus to the IXR, let us consider a single dual-polarisation antenna characterised by Jones matrix $\mathbf{J}$, i.e., $\mathbf{V} = \mathbf{J}$, and noise covariance matrix $\mathbf{R}_\mathrm{n} = \sigma_\mathrm{n}\mathbf{I}$, where σ_n is the noise power in each receive path. Substitution of Eq. (8) in Eq. (4) gives

$$\mathbf{C}_\mathrm{opt} = \mathbf{V}^H\mathbf{R}_\mathrm{n}^{-1}\mathbf{V}. \qquad (11)$$

For the scenario in our example, this becomes

$$\mathbf{C}_\mathrm{opt} = \mathbf{J}^H\sigma_n^{-1}\mathbf{I}\mathbf{J} = \sigma_n^{-1}\mathbf{J}^H\mathbf{J}. \qquad (12)$$

Since the largest singular value of $\mathbf{J}$ is equal to the square root of the largest eigenvalue of $\mathbf{J}^H\mathbf{J}$ and a similar relation holds between the smallest singular value of $\mathbf{J}$ and the smallest eigenvalue of $\mathbf{J}^H\mathbf{J}$, we have

$$\mathrm{SR} = \kappa\left(\mathbf{C}_\mathrm{opt}\right) = \kappa^2(\mathbf{J}). \qquad (13)$$

Note that the scaling with σ_n^{-1} does not affect this relationship as a scaling of all eigenvalues cancels in the ratio of those eigenvalues.

5 Example: non-diagonal noise covariance matrix

To illustrate the significance of the assumption that $\mathbf{R}_\mathrm{n} = \sigma_\mathrm{n}\mathbf{I}$, i.e., that the noise is spatially white, let us consider an antenna consisting of two orthogonally-placed ideal dipoles. The jones matrix for this antenna can be described by

$$\mathbf{J} = \begin{bmatrix} \cos\phi\cos\theta & -\sin\phi \\ \sin\phi\cos\theta & \cos\phi \end{bmatrix}, \quad (14)$$

where θ and ϕ denote the boresight angle and azimuthal angle respectively. To construct a spatially non-white noise covariance matrix, let us assume the following model for $\mathbf{R}_\mathrm{n}$:

$$\mathbf{R}_n = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}, \quad (15)$$

where the parameter α can be used to tune the strength of the effect of correlated noise between the two feeds. Note that both $\mathbf{J}$ and $\mathbf{R}_\mathrm{n}$ require scaling factors to describe a system with physically meaningful antenna gains and noise powers. Since these factors will cancel once we calculate gain or sensitivity ratios, we can ignore these factors in our analysis for convenience of notation.

Substitution of Eqs. (14) and (15) in Eq. (11) gives

$$\mathbf{C}_\mathrm{opt} = \frac{1}{1-\alpha^2} \times \begin{bmatrix} \cos^2\theta - \alpha\sin 2\phi\cos^2\theta & -\alpha\cos 2\phi\cos\theta \\ -\alpha\cos 2\phi\cos\theta & 1+\alpha\sin 2\phi \end{bmatrix}. \quad (16)$$

This shows that the eigenvalues of $\mathbf{C}_\mathrm{opt}$, and hence their ratio, are a function of azimuth and boresight angle. When the noise is spatially white, $\alpha = 0$ and the azimuthal dependence vanishes. To obtain bounds on the SR when $\alpha \neq 0$, we can solve the characteristic polynomial in closed form, and differentiate $(\lambda_\mathrm{max} - \lambda_\mathrm{min})^2$ with respect to ϕ to find the azimuthal angle for which the difference between the two eigenvalues is maximised or minimised. This happens when $\cos 2\phi = 0$, i.e., when $\sin 2\phi = \pm 1$. This simplifies the expression for $\mathbf{C}_\mathrm{opt}$ to

$$\mathbf{C}_\mathrm{opt} = \frac{1}{1-\alpha^2} \begin{bmatrix} (1\mp\alpha)\cos^2\theta & 0 \\ 0 & 1\pm\alpha \end{bmatrix}. \quad (17)$$

This gives the characteristic polynomial

$$\left((1\mp\alpha)\cos^2\theta - \lambda\right)\left((1\pm\alpha) - \lambda\right) = 0, \quad (18)$$

from which the following eigenvalues are obtained:

$$\begin{aligned} \lambda_1 &= 1\pm\alpha \\ \lambda_2 &= (1\mp\alpha)\cos^2\theta \end{aligned} \quad (19)$$

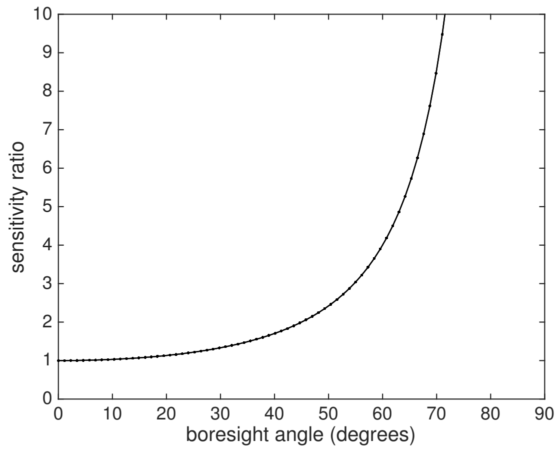

Figure 1. Sensitivity ratio as function of boresight angle for $\alpha = 0$. The dotted black curve shows $\kappa(\mathbf{J})^2$ while the solid black curves show the upper and lower bound for the variation in SR over azimuth (coinciding in this case).

This implies that the SR varies as function of azimuth between lower bound

$$\mathrm{SR}_\mathrm{min} = \max\left\{\frac{(1+\alpha)\cos^2\theta}{1-\alpha}, \frac{1-\alpha}{(1+\alpha)\cos^2\theta}\right\} \quad (20)$$

and upper bound

$$\mathrm{SR}_\mathrm{max} = \frac{1+\alpha}{(1-\alpha)\cos^2\theta}. \quad (21)$$

Note that both bounds depend on the value of α. The IXR, on the other hand, is insensitive to the level of spatial non-whiteness of the noise covariance matrix as it is computed based $\kappa(\mathbf{J})$, which, using Eq. (13), follows from

$$\kappa(\mathbf{J}) = \sqrt{\kappa\left(\mathbf{C}_\mathrm{opt}\Big|_{\alpha=0}\right)} = \frac{1}{|\cos\theta|}. \quad (22)$$

Note that the upper and lower bound for the SR converge to the square of this value when α converges to 0. This allows us to relate the SR to the IXR.

To validate the findings above, this scenario was simulated numerically. The results are shown in Figs. 1 and 2 for $\alpha = 0$ and $\alpha = 0.2$ respectively. In both plots, the dotted black curve shows $\kappa(\mathbf{J})^2$, which gives the SR for the case of spatially-white noise, while the solid black curves show the upper and lower bound for the SR. Between the bounds, there are many grey curves showing the SR for a series of azimuthal angles between 0 and 2π. In Fig. 1, all curves coincide, as expected. Figure 2 shows that the introduction of spatially non-white noise has a significant impact on system performance. Towards boresight, the upper and lower bound for the SR are the same, but both are higher, i.e., worse, than when $\alpha = 0$. At higher boresight angles, in particular above 25°, the sensitivity ratio is better than one would expect purely based on the antenna response to the external signal for some azimuthal angles while being

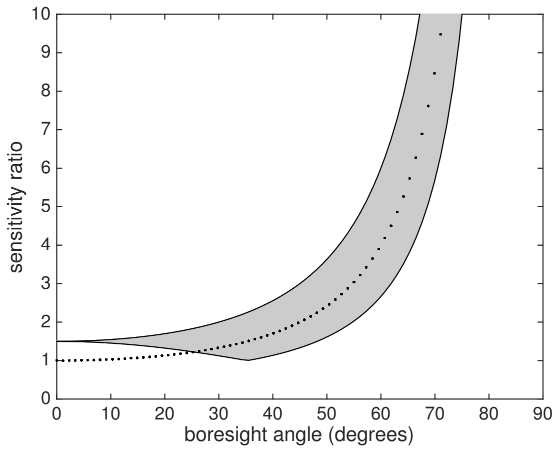

Figure 2. Sensitivity ratio as function of boresight angle for $\alpha = 0.2$. The dotted black curve shows $\kappa(\mathbf{J})^2$ while the solid black curves show the upper and lower bound for the variation in SR over azimuth and the grey curves show the actual SR at specific azimuth angles.

worse at other azimuthal angles. Depending on what is considered to be an acceptable sensitivity ratio, the boresight angle at which the threshold is passed can vary by over 15° as function of azimuth angle. This example therefore illustrates that the presence of correlated noise between receive paths may have a significant impact on the scan range over which the antenna (array) meets the desired performance.

6 Conclusions

Current figures-of-merit (FoMs) for the polarimetric performance of an antenna (array) are purely based on the far-field response of the antenna (array). These FoMs therefore neglect the effect that spatially non-white noise (e.g., due to inhomogeneous background noise or noise coupling) may have on the reconstruction of the polarisation state of the received signals. To overcome this drawback, a new FoM for polarimetric performance, the sensitivity ratio (SR) is introduced. The SR is defined as the ratio of the sensitivity of the antenna (array) to a signal with the most favourable polarisation and a signal of identical power with the least favourable polarisation. The SR is independent of the coordinate system used to describe the measurement system, can account for any noise covariance structure and can be computed for individual antennas as well as beamformed arrays. For spatially white noise, we have a simple relation between SR and IXR. With a numerical example amenable to closed-form analysis the possible significance of spatially non-white noise was illustrated.

7 Acknowledgements

The author would like to thank Wim van Cappellen and Jaap Bregman for their useful comments on the first draft of this paper. This work was supported by the Netherlands Organisation for Scientific Research.

References

[1] *IEEE Standard Definitions of Terms for Radio Wave Propagation*, IEEE Std. 211-1997, 1997.

[2] T. D. Carozzi and G. Woan, "A Fundamental Figure of Merit for Radio Polarimeters," *IEEE Transactions on Antennas and Propagation*, vol. 59, no. 6, pp. 2058–2065, Jun. 2011.

[3] S. J. Wijnholds and W. A. van Cappellen, "In Situ Antenna Performance Evaluation of the LOFAR Phased Array Radio Telescope," *IEEE Transactions on Antennas and Propagation*, vol. 59, no. 6, pp. 1981–1989, Jun. 2011.

[4] R. Maaskant and E. E. M. Woestenburg, "Applying the active antenna impedance to achieve noise match in receiving array antennas," in *2007 IEEE Antennas and Propagation Society International Symposium*, Honolulu (HI), USA, 9-15 Jun. 2007.

[5] M. V. Ivashina, R. Maaskant, and B. Woestenburg, "Equivalent System Representation to Model the Beam Sensitivity of Receiving Antenna Arrays," *IEEE Antennas and Wireless Propagation Letters*, vol. 7, pp. 733–737, Oct. 2008.

[6] K. F. Warnick, M. V. Ivashina, S. J. Wijnholds, and R. Maaskant, "Polarimetry With Phased Array Antennas: Theoretical Framework and Definitions," *IEEE Transactions on Antennas and Propagation*, vol. 60, no. 1, pp. 184–196, Jan. 2012.

[7] R. Horn and C. Johnson, *Topics in Matrix Analysis*. Cambridge, UK: Cambridge University Press, 1994.

[8] M. V. Ivashina, O. Iupikov, R. Maaskant, W. A. van Cappellen, and T. Oosterloo, "An optimal beamforming strategy for wide-field surveys with phased-array-fed reflector antennas," *IEEE Transactions on Antennas and Propagation*, vol. 59, no. 6, pp. 1864–1875, Jun. 2011.

[9] W. A. van Cappellen and S. J. Wijnholds, "Polarimetric Calibration Results of an APERTIF Phased Array Feed," in *International Conference on Electromagnetics in Advanced Applications (ICEAA)*, Cape Town, South Africa, 2-7 Sep. 2012.

[10] G. Virone *et al.*, "Strong Mutual-Coupling Effects on LOFAR: Modeling and In-Situ Validation," *IEEE Transactions on Antennas and Propagation*, 2018, submitted after revision.